SIMPLEX ALGORITHMS

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1. Standard Equality Form

Definition 1.1 (SEF)**.** An LP is said to be in *standard equality form* (SEF) if it is of the form

$$\max\{\mathbf{c}^{\mathsf{T}}\mathbf{x} + \bar{z} \mid \mathbf{A}\mathbf{x} = b, \mathbf{x} \geq \mathbf{0}\}$$

where $\bar{z}$ denotes some constant.

Example 1.1 (SEF)**.**

$$\begin{aligned} \max \quad & \begin{bmatrix} 2 & 3 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} - 4 \\ \text{s.t.} \quad & \\ & \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 6 \\ 10 \\ 4 \end{bmatrix} \\ & \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5 \geq \mathbf{0} \end{aligned}$$

How to make a problem into SEF?

- Convert **min**inimzation into **max**inimzation by replacing $\max f(x)$ by $-z$ where $z = \min[-f(x)]$.
- Deal with free variables. Say x_2 is a free variable (no constraint $x_2 \geq 0$ is imposed). Here, we replace x_2 by $x_2' - x_2''$ where $x_2', x_2'' \geq 0$.
- Deal with inequality constraints. Say $x_1 + x_2 \geq 4$. There is a *surplus.* Let $s \geq 0$. Then we replace the inequality constraint by $x_1 + x_2 - s = 4$. Likewise, if $x_1 + x_2 \leq 4$. We replace it by $x_1 + x_2 + s = 4$ and here s is called a *slack variable.*

2. Bases

Consider an $m \times n$ matrix $\mathbf{A}$, where the rows of $\mathbf{A}$ are linearly independent.

Definition 2.1. Denote the column j of $\mathbf{A}$ by $\mathbf{A}_j$.

Example 2.1. Let $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \end{bmatrix}$. Then $\mathbf{A}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Definition 2.2. Let $J \subseteq \{1, \ldots, n\}$. Define $\mathbf{A}_J$ to be the matrix formed by columns $\mathbf{A}_j$ for all $j \in J$.

Example 2.2. Let $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \end{bmatrix}$ and $B = \{1, 3, 5\}$.

Then, $\mathbf{A}_B = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Definition 2.3 (Basis)**.** We say that a given set of column indices of B form a *basis* if the matrix $\mathbf{A}_B$ is a **square non-singular** matrix.

Example 2.3. Let $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \end{bmatrix}$. $B = \{3, 4, 5\}$ is a basis, because $\mathbf{A}_B = I$ is square non-singular.

Definition 2.4. Denote by N the set of column indices not in B.

Example 2.4. Let $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \end{bmatrix}$ and $B = \{3, 4, 5\}$.

Then, $N = \{1, 2\}$

Proposition 2.1. *B and N denote a partition of the column indices of $\mathbf{A}$.*

Definition 2.5. Variables $\mathbf{x}_j$ are said to be *basic* when $j \in B$ and *nonbasic* otherwise.

Example 2.5. Let $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \end{bmatrix}$. If $B = \{3, 4, 5\}$, then variables $\mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5$ are basic.

Proposition 2.2.

$$\mathbf{A}\mathbf{x} = \sum_{j=1}^{n} \mathbf{x}_j \mathbf{A}_j = \sum_{j \in B} \mathbf{x}_j \mathbf{A}_j + \sum_{j \in N} \mathbf{x}_j \mathbf{A}_j = \mathbf{A}_B \mathbf{x}_B + \mathbf{A}_N \mathbf{x}_N$$

Example 2.6. Let $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \\ \mathbf{x}_4 \\ \mathbf{x}_5 \end{bmatrix}$. If $B = \{3, 4, 5\}$, then $N = \{1, 2\}$. Note that

$$\mathbf{A}\mathbf{x} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \\ \mathbf{x}_4 \\ \mathbf{x}_5 \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 \\ 2\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_4 \\ -\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_5 \end{bmatrix}$$

$$\mathbf{A}_B \mathbf{x}_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_3 \\ \mathbf{x}_4 \\ \mathbf{x}_5 \end{bmatrix} = \begin{bmatrix} \mathbf{x}_3 \\ \mathbf{x}_4 \\ \mathbf{x}_5 \end{bmatrix}$$

$$\mathbf{A}_N \mathbf{x}_N = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1 + \mathbf{x}_2 \\ 2\mathbf{x}_1 + \mathbf{x}_2 \\ -\mathbf{x}_1 + \mathbf{x}_2 \end{bmatrix}$$

$$\mathbf{A}_B \mathbf{x}_B + \mathbf{A}_N \mathbf{x}_N = \begin{bmatrix} \mathbf{x}_3 \\ \mathbf{x}_4 \\ \mathbf{x}_5 \end{bmatrix} + \begin{bmatrix} \mathbf{x}_1 + \mathbf{x}_2 \\ 2\mathbf{x}_1 + \mathbf{x}_2 \\ -\mathbf{x}_1 + \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 \\ 2\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_4 \\ -\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_5 \end{bmatrix} = \mathbf{A}\mathbf{x}$$

Definition 2.6 (Basic Solution)**.** A vector $\bar{\mathbf{x}}$ is a *basic solution* of $\mathbf{A}\mathbf{x} = \mathbf{b}$ for a basis B if the following conditions hold:

(1) $\mathbf{A}\bar{\mathbf{x}} = \mathbf{b}$ and
(2) $\bar{\mathbf{x}}_N = \mathbf{0}$

Theorem 2.1. Every basis is associated with a unique basic solution.

Proof. Suppose $\bar{\mathbf{x}}$ is a basic solution, then

$$\mathbf{b} = \mathbf{A}\bar{\mathbf{x}} = \mathbf{A}_B\bar{\mathbf{x}}_B + \mathbf{A}_N \underbrace{\bar{\mathbf{x}}_N}_{=\mathbf{0}} = \mathbf{A}_B\bar{\mathbf{x}}_B$$

Since $\mathbf{A}_B$ is non-singular, it has an inverse and we have

$$\bar{\mathbf{x}}_B = \mathbf{A_B}^{-1}\mathbf{b}$$

□

Example 2.7. Consider the LP

$$\begin{array}{ll} \max & \begin{bmatrix} 2 & 3 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} - 4 \\ \text{s.t.} & \\ & \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 6 \\ 10 \\ 4 \end{bmatrix} \\ & \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5 \geq \mathbf{0} \end{array}$$

If $B = \{3, 4, 5\}$, then the unique basic solution $\bar{\mathbf{x}}$ is $\bar{\mathbf{x}}_1 = \bar{\mathbf{x}}_2 = \mathbf{0}$ and

$$\begin{aligned} \bar{\mathbf{x}}_B &= \mathbf{A_B}^{-1}\mathbf{b} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 10 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 6 \\ 10 \\ 4 \end{bmatrix} \end{aligned}$$

Thus, the unique basic solution $\bar{\mathbf{x}}$ is $\bar{\mathbf{x}} = \begin{bmatrix} 0 \\ 0 \\ 6 \\ 10 \\ 4 \end{bmatrix}$.

Definition 2.7. A basic solution $\bar{\mathbf{x}}$ is *feasible* if $\bar{\mathbf{x}} \geq \mathbf{0}$.

Example 2.8. The basic solution in Example 2.7 is feasible since $\bar{\mathbf{x}} \geq \mathbf{0}$.

Definition 2.8. A basis B is *feasible* if the corresponding basic solution is feasible. If a basic solution is not feasible, then it is infeasible.

Example 2.9. In Example 2.7, the basis $B = \{3, 4, 5\}$ is feasible, since the corresponding basic solution is feasible as in Example 2.8.

3. Canonical Forms

Definition 3.1. A LP in SEF is in canonical form for a basis B if it satisfies the following two conditions:

(1) $\mathbf{A}_B = \mathbf{I}$
(2) $\mathbf{c}_B = \mathbf{0}$

Conditions (1) and (2) are refered to as C1 and C2, respectively.

Example 3.1. Consider the LP

$$\begin{aligned} \max \quad & \begin{bmatrix} 2 & 3 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} - 4 \\ \text{s.t.} \quad & \\ & \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 6 \\ 10 \\ 4 \end{bmatrix} \\ & \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5 \geq \mathbf{0} \end{aligned}$$

It is in canonical form for the basis $B = \{3, 4, 5\}$.

For any given basis B, we can find a canonical form for it. Let (P) be in SEF as follows:

$$\max\{\mathbf{c}^\intercal \mathbf{x} + \bar{z} \mid \mathbf{A}\mathbf{x} = b, \mathbf{x} \geq \mathbf{0}\}$$

where $\bar{z}$ denotes some constant.

The conversion involves two steps, and they are described and justified by the following claims:

(1) Since B is a basis, $\mathbf{A}_B$ is invertible. We claim that to achieve $C1$, it suffices to multiply $\mathbf{A}_B^{-1}$ on the left sides of $\mathbf{A}\mathbf{x} = \mathbf{b}$.
(2) We claim that to achieve $C2$, it suffices to replace $\mathbf{c}^\intercal \mathbf{x} + \bar{z}$ by $\mathbf{y}^\intercal \mathbf{b} + \bar{z} + (\mathbf{c}^\intercal - \mathbf{y}^\intercal \mathbf{A})\,\mathbf{x}$, where $\mathbf{y}$ equals $(\mathbf{A}_B^\intercal)^{-1}\,\mathbf{c}_B$.

Now let's prove the claims.

Proof of claim (1). The term we will replace is $\mathbf{Ax} = \mathbf{b}$. Since $\mathbf{A}_B^{-1}$ exists, multiplying $\mathbf{A}_B^{-1}$ on the left sides of $\mathbf{Ax} = b$, we get $\mathbf{A}_B^{-1}\mathbf{Ax} = \mathbf{A}_B^{-1}\mathbf{b}$. By Proposition 2.2, we know that $\mathbf{Ax} = \mathbf{A}_B\mathbf{x}_B + \mathbf{A}_N\mathbf{x}_N$. $\mathbf{A}_B^{-1}\mathbf{Ax} = A_B^{-1}\left(\mathbf{A}_B\mathbf{x}_B + \mathbf{A}_N\mathbf{x}_N\right)$. Simplify and get $\mathbf{A}_B^{-1}\mathbf{Ax} = \mathbf{x}_B + \mathbf{A}_B^{-1}\mathbf{A}_N\mathbf{x}_N$. □

Proof of claim (2). This is concerning the change in the objective function. Let B be a basis of A, where A has m rows. Let $\mathbf{y}^\intercal = \begin{bmatrix} y_1 & \cdots & y_m \end{bmatrix}$. If $\bar{\mathbf{x}}$ is a feasible solution, then $\mathbf{A}\bar{\mathbf{x}} = \mathbf{b}$. Multiplying $\mathbf{y}^\intercal$ on left sides of $\mathbf{y}^\intercal\mathbf{A}\bar{\mathbf{x}} = \mathbf{y}^\intercal\mathbf{b}$, we get $\mathbf{y}^\intercal\mathbf{A}\bar{\mathbf{x}} = \mathbf{y}^\intercal\mathbf{b}$, which is equivalent to $0 = \mathbf{y}^\intercal\mathbf{b} - \mathbf{y}^\intercal\mathbf{A}\bar{\mathbf{x}}$. Every feasible solution $\bar{\mathbf{x}}$ will satisfy $0 = \mathbf{y}^\intercal\mathbf{b} - \mathbf{y}^\intercal\mathbf{A}\bar{\mathbf{x}}$. Thus, we may add $\mathbf{y}^\intercal\mathbf{b} - \mathbf{y}^\intercal\mathbf{A}\bar{\mathbf{x}}$ to the objective function (without changing the value of it). The original objective function is $\mathbf{c}^\intercal\mathbf{x} + \bar{z}$. Hence, the objective function now is $\mathbf{c}^\intercal\mathbf{x} + \bar{z} + \mathbf{y}^\intercal\mathbf{b} - \mathbf{y}^\intercal\mathbf{A}\bar{\mathbf{x}}$. Rearrange and get $\mathbf{y}^\intercal\mathbf{b} + \bar{\mathbf{z}} + \left(\mathbf{c}^\intercal - \mathbf{y}^\intercal\mathbf{A}\right)\mathbf{x}$. Let $\bar{\mathbf{c}}^\intercal = \mathbf{c}^\intercal - \mathbf{y}^\intercal\mathbf{A}$. To achieve $C2$, we need $\bar{\mathbf{c}}_B = \mathbf{0}$, which is equivalent to $\mathbf{c}_B^\intercal - \mathbf{y}^\intercal\mathbf{A}_B = \mathbf{0}$. Rearrange and get $\mathbf{y}^\intercal\mathbf{A}_B = \mathbf{c}_B^\intercal$. Transpose both sides and get $\mathbf{A}_B^\intercal y = \mathbf{c}_B$. Since the rows of A are linearly independent, $\left(\mathbf{A}_B^\intercal\right)^{-1}$ exists. Multiply $\left(\mathbf{A}_B^\intercal\right)^{-1}$ on both sides of $\mathbf{A}_B^\intercal y = \mathbf{c}_B$ and get $\mathbf{y} = \left(\mathbf{A}_B^\intercal\right)^{-1}\mathbf{c}_B$. This $\mathbf{y}$ is what we construct in the first place. □

Simplex Algorithms change from one canonical form to another by applying the replacements in the above claims. Eventually, at times we obtain a canonical form where the objective function has no positive value, the algorithm shall terminate.