CO 342 Graph Theory

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Contents

Lecture 1

Introduction to Matching and Greedy Algorithm for Matching

1.1. Basic Definitions

Definition 1.1 (Matching)**.** A *matching* in a graph G is a set of edges, no two of which share a vertex.

Definition 1.2. A matching M saturates a vertex v if v is incident to an edge of M.

Definition 1.3 (Perfect Matching)**.** A matching M in a graph G is perfect, if M saturates all vertices in $V(G)$.

Remark 1.4. An obvious consequence from Definition 1.3 is that if G has a perfect matching, then $|V(G)|$ is even.

We will see a characterization of graphs with perfect matchings (Tutte's theorem) and an algorithm for finding a maximum matching in general graphs (Edmond's algorithm). Interestingly, both of them were here in our university.

1.2. Greedy Algorithm for Matching

Input: Graph G. **Output:** A matching of G.

1. $M := \emptyset$, $H := G$.
2. If H has no edges, stop and output M.
3. Pick an edge xy in H. Set $M := M \cup \{xy\}$.
4. Set $H := H - \{xy\}$, and go to step 2.

Definition 1.5. We denote $\nu(G)$ the size of a maximum matching in G.

Theorem 1.6. *The greedy algorithm finds a matching in G of size at least* $\frac{|\nu(G)|}{2}$.

Proof. Let M be a matching in a graph G obtained by the Greedy Algorithm. Then each edge of G shares a vertex with an edge of M. Let M' be a maximum matching in G. Then, each edge of M can share a vertex with ≤ 2 edges of M'. Thus, $|M'| \leq 2|M|$. Hence, $|M| \geq \frac{\nu(G)}{2}$. □

Lecture 2

Hall's Theorem and its Defect Version

Definition 2.1 (Neighbourhood)**.** For a set S of vertices of a graph G, the neighbourhood of S is $\Gamma(S) = \{y \in V(G) : xy \in E(G) \text{ for some } x \in S\}$.

Theorem 2.2 (Hall's Theorem)**.** *Let G be a bipartite graph with vertex classes X and Y, then G has a matching saturating X if and only if*

$$(\star) \quad |\Gamma(S)| \geq |S| \quad \forall S \subseteq X$$

We call $(\star)$ *Hall's Condition*.

Proof. If G has a matching saturating X, then for every S just the set of neighbours of S via edges M has size $|S|$, so $(\star)$ holds.

Conversely, we assume $(\star)$ holds. By induction on $|X|$. If $|X| = 1$, then the statement is clearly true. We may assume that $|X| \geq 2$ and statement holds for any graph with smaller X.

- Case 1: Suppose $|\Gamma(S)| > |S|$ for every $\emptyset \neq S \neq X$. Let $x \in X$ be arbitrary. Choose a neighbour y of X in Y, which is possible because $(\star)$ holds. Let $H = G - \{x, y\}$. Then $(\star)$ holds for H, since

  $$|\Gamma_H(S)| \geq |\Gamma_G(S)| - 1 \geq |S|$$

  Then H has a matching M_H saturating $X - \{x\}$, by induction hypothesis. Thus, $M_H \cup \{xy\}$ is a matching in G saturating X.

□

Lecture 3

Application to Hall's Theorem and Revisit of König's Theorem

3.1. Application to Hall's Theorem

Theorem 3.1. *Any regular bipartite graph with degree $k \geq 1$ has a perfect matching.*

Proof. Let X and Y be the vertex classes of G. Let $S \subseteq X$ be an arbitrary subset of X. Let $E(S, \Gamma(S))$ denote the set of edges of G from S to $\Gamma(S)$. Then,

$$|E(S, \Gamma(S))| = k|S|.$$

On the other hand, note that

$$|E(S, \Gamma(S))| \leq k|\Gamma(S)|,$$

since G is k-regular.

Hence, $k|S| \leq k|\Gamma(S)| \implies |\Gamma(S)| \geq |S|$, so Hall's Theorem implies G has a matching of size $|X|$. Similarly, it has a matching of size $|Y|$. Hence, $|X| = |Y|$ and G has a perfect matching. □

3.2. König's Theorem

Recall $\nu(G)$ is the maximum size of a matching in G.

Definition 3.2 (Vertex Cover)**.** A vertex cover of a graph G is a set $C \subseteq V(G)$ such that every edge of G is incident to a vertex of C.

We denote by $\tau(G)$ the minimum size of a vertex cover of G.

Note that

$$\nu(G) \leq \tau(G) \leq 2\nu(G)$$

holds for every graph. Since all vertices saturated by a maximal matching form a vertex cover. Note the equality $\tau(G) = 2\nu(G)$ holds for K_3.

Theorem 3.3 (König's Theorem)**.** *If G is a bipartite graph, then $\tau(G) = \nu(G)$.*

Proof. Let X and Y denote the vertex classes of G. Construct H by adding new vertices x and y and edges $\{xz : z \in X\}$ and $\{yz : z \in Y\}$. Note that if C is a vertex cut of H separating x and y in H, then C must contain a vertex cover of G; otherwise, a path from x to y would remain in $H - C$. If S is a set of internally disjoint (x, y)-paths in H, then the set consisting of the second edges of all paths in S forms a matching in G. Therefore, $\tau(G) \leq$ *(the minimum size of vertex cut separating x and y in H)* $=$ *(the maximum size of a set of internally disjoint (x, y)-paths in H)* , by Menger's theorem. □

Definition 3.4. Let M be a matching in a graph G. An M-alternating path is a path in G with every second edge in M.

Definition 3.5. We say a vertex v is M-exposed if it is not M-saturated.

Definition 3.6. An M-augmenting path is an M-alternating path of length ≥ 1 that starts and ends with an M-exposed vertex.

If P is an M-augmenting path in G, then $M' = M\Delta E(P) = M \setminus (M \cap E(P)) \cup (E(P) \setminus M)$ is also a matching in G with $|M'| = |M| + 1$. We sometimes say M' is "M switched on P".

Theorem 3.7. *(Berge's Theorem) A matching M in a graph G is a maximum matching if and only if there is no M-augmenting path in G.*

Proof. If M is maximum, then there is no M-augmenting path P; otherwise, we have that $M\Delta E(P)$ contradicts the maximality of M. Conversely suppose there is no M-augmenting path in G. Let M^* be a maximum matching in G.Consider the subgraph H of G with edge set $M \cup M^*$. The components of H can be

1. single edges in $M \cap M^*$,
2. even cycles that are (M, M^*)-alternating, or
3. paths that are (M, M^*)-alternating.

Note that no path component can have more M^*-deges than M-edges, since it would be an M-augmenting path. Therefore, the number of M^* edges is at most the number of M-edges in every type of components of H. Hence, $|M^*| \leq |M|$ implies that M is maximum. □

Lecture 4

Independent Set and Edge Cover

Definition 4.1 (Independent Set)**.** An independent set of vertices in a graph G is a set $W \subseteq V(G)$ such that $G[W]$ induced by W has no edges.

We denote by $\alpha(G)$ the maximum size of an independent set of vertices in G.

Definition 4.2 (Edge Cover)**.** An edge cover of a graph G is a set $S \subseteq E(G)$ such that every vertex of G is incident to an edge of S.

We denote by $\rho(G)$ the minimum size of an edge cover of G.

Remark 4.3. If G has an isolated vertex, $\rho(G) = \infty$ or we say $\rho(G)$ is undefined.

Lemma 4.4. *For every graph G,*

$$\alpha(G) + \tau(G) = |V(G)|.$$

Proof. Any vertex cover C is such that $G - C$ has no edges. The subgraph induced by $V(G) \setminus C$ is an independent set. C is a minimum vertex cover if and only if $V(G) - C$ is a maximum independent set. □

Lemma 4.5 (Gallai Lemma)**.** *Let G be a graph with no independent vertices, then*

$$\nu(G) + \rho(G) = |V(G)|$$

Proof. Let $|V(G)| = n$ and let M be a maximum matching (so $|M| = \nu(G)$). Let $V(M)$ be the set of vertices saturated by M. Then, $V(G) \setminus V(M)$ is independent. Form an edge cover S of G by taking one edge incident to each $x \in V(G) \setminus V(M)$, together with the matching edges of M. This gives $n - 2|M| + |M| = n - |M| = n - \nu(G)$ edges in S. $\rho(G) \leq |S| = n - \nu(G) \implies \rho(G) + \nu(G) \leq n$.

Conversely, suppose F is an edge cover of G of size $\rho(G)$. Then each edge of F is incident to a vertex that is not incident to any other edge of F. Therefore, the spanning subgraph H of G with $E(H) = F$ has the property that every component is a star with at least 1 edge. Thus, H has no cycles. Then, there is a matching in G formed by taking an edge from each component of H. Since H has n vertices and $\rho(G)$ edges, and no cycles, it has exactly $n - \rho(G)$ components. Thus, $\nu(G) \geq n - \rho(G)$. (This result follows from Math 239.)

Hence, $\rho(G) + \nu(G) = n$. □

Theorem 4.6 (Erdös-Posa)**.** *For every graph G,*

$$\nu(G) \geq \min\left\{\delta(G), \left\lfloor \frac{|V(G)|}{2} \right\rfloor\right\}$$

Proof. Let M be a maximum matching and let $V(M)$ denote the set of vertices saturated by M. If $|M| \geq \left\lfloor \frac{|V(G)|}{2} \right\rfloor$, we are done, so assume $|M| \leq \left\lfloor \frac{|V(G)|}{2} \right\rfloor - 1$, so $|V(M)| \leq |V(G)| - 2$. Let $x, y \in V(G) \setminus V(M)$. Then all edges incident to x or y are also incident to some vertex of $V(M)$. For $uv \in M$, note that if $xu \in E(G)$, then $yv \notin E(G)$; otherwise x and y are M-exposed, and we get an M-augmenting path, so the number of edges from $\{x, y\}$ to $\{u, v\}$ is at most 2. The total number N of edges from $\{x, y\}$ to $V(M)$ is at most $2|M|$. $d(x) + d(y) \leq 2|M| \implies 2\delta(G) \leq 2|M| \implies |M| \geq \delta(G)$. □

Lecture 5

Stable Matching in Bipartite Graphs

5.1. Stable Matching

Definition 5.1 (Preference List)**.** Let G be a bipartite graph with vertex classes X and Y. Suppose for each vertex z of G, there is a linear order $L(z)$ of vertices in $\Gamma(z)$. Here $L(z)$ is called the preference list for z.

Definition 5.2 (Stable Matching)**.** A stable matching in (G, L) is a matching M such that for every edge $xy \notin M$, either

- $xy' \in M$ for some $y' > y$ in $L(x)$, or
- $x'y \in M$ for some $x' > x$ in $L(y)$.

We will see that every (G, L) has a stable matching for bipartite graph G.

Remark 5.3. A stable matching is not necessarily a maximum matching.

5.2. Gale-Shapley Algorithm

Let G be a partitie graph with vertex classes X and Y.

Input: G and preference lists L. Output: A stable matching M^* in G.

1. Set $K(x) = L(x)$ for each $x \in X$. Set $M := \emptyset$.
2. If for each $x \in X$, either $K(x) = \emptyset$ or x is M-saturated, then STOP. Set $M^* := M$ and output M^*.
3. Otherwise, choose $x \in X$ where $K(x) \neq \emptyset$ and x is M-exposed. Let y be the largest element of $K(x)$, so $xy \in E(G)$.
4. If there exists $x' \in V(G)$ such that $x'y \in M$ and $x > x'$ in $L(y)$, then set $M := M \setminus \{x'y\} \cup \{xy\}$.
5. If y is M-exposed, then set $M := M \cup \{xy\}$.
6. Set $K(x) := K(x) \setminus \{y\}$.
7. Goto Step 2.

Remark 5.4. At all times in the algorithm, the situation for each $y \in Y$ only improves (or stays the same) and for each $x \in X$ only deteriorates (or stays the same).

Theorem 5.5. *The Gale-Shapley Algorithm finds a stable matching in* (G, L).

Proof. First note that the quantity $\sum_{x \in X} |K(x)|$ decreases by 1 at each iteration, so the algorithm terminates in at most $\sum_{x \in X} |L(x)| = |E(G)|$.

Too see M^* is a matching, note that in steps 4 and 5, if an edge xy is added to M, then x is M-exposed, and any edge incident to y is removed.

To show M^* is stable, consider an edge $x_0 y_0 \notin M^*$.

- Case 1: $y_0 = y$ at line 3 in some iteration I with $x_0 = x$. If $x_0 y_0$ is put into M in step 4, then at some later iteration I', it was removed. Thus, in I' there was $x_1 > x_0$ in $L(y)$, and $x_1 y_0$ was put into M. By Remark 5.4, $x_f y_0 \in M^*$ for some $x_f > x_0$ in $L(y_0)$. If $x_0 y_0$ is not put into M in I, then $x_1 y_0 \in M$ already for some $x_1 > x_0$ in $L(y_0)$.
- Case 2: y_0 is never considered at line 3 when $x = x_0$. Since $y_0 \in L(x_0)$ at termination, then x_0 is matched to some $y_1 > y_0$ in $L(x_0)$ at termination.

Hence, M is stable. □

Lecture 6

Stable Matching Cont'd

It is possible to have many stable matchings for a bipartite graph for fixed preference lists.

Theorem 6.1. *All stable matchings in (G, L) have the same size.*

Proof. Let M_1 and M_2 be two stable matchings in G with preference lists L. Let H be the subgraph of G with edge set $M_1 \cup M_2$. Then every component of H is a path or an even cycle.

If $|M_1| > |M_2|$, then some path component $P = x_1y_1x_2y_2 \dots x_ky_k$ must have more M_1-edges than M_2-edges. Thus, $x_iy_i \in M_1$ for each i, $y_ix_{i+1} \in M_2$ for each i and x_1 and y_k are M_2-exposed. Since $x_1y_1 \notin M_2$, and x_1 is M_2-exposed, in $L(y_1)$ we have $x_2 > x_1$. Since $x_2y_1 \notin M_1$, and $x_2 > x_1$ in $L(y_1)$, we find $y_2 > y_1$ in $L(x_2)$.

Suppose for $2 \leq i \leq k-1$, we know that $y_i > y_{i-1}$ in $L(x_i)$. Then since $x_iy_i \notin M_2$, we conclude $x_{i+1} > x_i$ in $L(y_i)$. Then since $x_{i+1}y_i \notin M_1$, we find $y_{i+1} > y_i$ in $L(x_{i+1})$, but then $y_k > y_{k-1}$ in $L(x_k)$ and $x_ky_k \notin M_2$. Since y_k is M_2-exposed,, which contradicts the fact that M_2 is stable.

Hence $|M_1| = |M_2|$. □

Definition 6.2 (X-Optimal)**.** Let G be a bipartite graph with vertex classes X and Y and preference lists L. A stable matching M is X-optimal if each $x \in X$ is matched by M to the best possible neighbour it could get in any stable matching.

Theorem 6.3. *The matching found by the Gale-Shapley algorithm is X-optimal.*

Proof. Let M' be an arbitrary stable matching in G and let M^* be the Gale-Shapley matching. Suppose on the contrary that $x_0y_0 \in M'$ for some $x_0 \in X$ where x_0 is either not M^*-exposed, or $X_0y^* \in M^*$ for some $y^* < y_0$ in $L(x_0)$.

Let I be the first iteration of Gale-Shapley algorithm such that

- there exists $x_0y_0 \in M'$ where x_0 is worse off in M^* and

- an edge y_0x_1 is put into M in iteration I where $x_1 > x_0$ in $L(y_0)$

Since $x_1y_0 \notin M'$ and M' is stable, for some y_1, we have $x_1y_1 \in M'$ for some $y_1 > y_0$ in $L(x_1)$. Then, in some earlier iteration I' of Gale-Shapley algorithm, x_1 proposed to y_1 and was rejected, because x_2y_1 was already in M for some $x_2 > x_1$ by y_1. Thus, in iteration I' we had:

- there exists $x_1y_1 \in M'$ where x_1 is worse off in M^* (recall situation only deteriorates for x in Gale-Shapley algorithm) and
- y_1x_2 is put into M in iteration I' where $x_2 > x_1$ in $L(y_1)$.

This contradicts our choice of I. Hence, M^* is X-optimal. □

Lecture 7

Consequences of the Gale-Shapley Algorithm and Tutte's Theorem

7.1. Consequences of the Gale-Shapley Algorithm

Similarly to Theorem 6.3, the Gale-Shapley algorithm finds the Y-pessimal stable matching in (G, L): for each $y \in Y$, either y is unmatched by the Gale-Shapley matching, or is matched to the worst possible neighbour it could get in any stable matching.

Corollary 7.1. *Every stable matching in (G, L) saturates the same set of vertices on X and Y.*

Proof. Suppose $x \in X$ is not saturated by the Gale-Shapley matching. We call the subset of X of unmatched vertices in the Gale-Shapley matching X_0. Since the Gale-Shapley algorithm is X-optimal, by Theorem 6.3, each $x \in X_0$ is unmatched in every stable matching. Since every stable matching M has the same size $|X| - |X_0|$, by Theorem 6.1, we find that M saturates exactly the set $X - X_0$. Similarly, $y \in Y$, if y is saturated by the Gale-Shapley matching, then since the Gale-Shapley algorithm is Y-pessimal, y is saturated by every stable matching. □

Corollary 7.2. *The matching found by the Gale-Shapley matching algorithm is independent of the order in which the "proposal" steps are executed.*

Proof. The X-optimal matching is unique. □

Stable Roommates Problem

Stable matchings in general (i.e. not bipartite) graphs: each vertex has a linear order list on its neighbours and a matching M is stable if for each edge $xy \notin M$, either $xy_1 \in M$ for some $y_1 > y$ in $L(x)$, or $x_1 y \in M$ for some $x_1 > x$ in $L(y)$.

Note that a stable matching in this setting might not exist.

7.2. Tutte's Theorem

A natural obstruction for the existence of a perfect matching in bipartite graphs is when the Hall's Condition does not hold, but when does a graph have a perfect matching in general?

Let G be a graph with a perfect matching M, and then in particular $|V(G)|$ is even. Suppose $T \subseteq V(G)$. Consider the components of $G - T$ (the subgraph of G induced by $V(G) \setminus T$). If C is an odd component with odd $|V(G)|$ of $G - T$, at least one edge of M must join the C to T. For each odd component, there is such an edge of M. Therefore, the number of odd components in $G - T$ is at most $|T|$.

We denote by $\text{odd}(H)$ the number of odd components in the graph H. Then, if G has a perfect matching, then for every $T \subseteq V(G)$, $\text{odd}(G - T) \leq |T|$.

Theorem 7.3 (Tutte's Theorem)**.** *A graph G has a perfect matching if and only if*

$$(\star\star) \qquad \forall\, T \subseteq V(G) \quad \text{odd}(G - T) \leq |T|$$

We call $(\star\star)$ the Tutte's Condition.

Remark 7.4. If G has an odd number of vertices, then it fails $(\star\star)$ when $T = \emptyset$, since G must have a component, e.g. itself if it is connected, that has an odd number of vertices.

Lemma 7.5 (Lemma A)**.** *Let G be a graph satisfying $(\star\star)$. If H is a graph with $V(H) = V(G)$ and $E(G) \subseteq E(H)$ then H satisfies $(\star\star)$.*

Proof. Let $T \subseteq V(G)$. Adding a new edge cannot increase the number of odd components (consider the cases). $\square$

Lecture 8

Proving Tutte's Theorem

Definition 8.1 (Type-0)**.** Let G be a graph. Let X be the set of vertices $x \in V(G)$ such that $\Gamma(x) = V(G) \setminus \{x\}$. Let $Y = V(G) \setminus X$. We say that G is type-0 if every component of the graph $G[Y]$ is a complete graph. (Note that X or Y could be empty.)

Lemma 8.2 (Lemma B)**.** *Let G be a graph satisfying*

$$(\star\star) \qquad \forall\, T \subseteq V(G) \quad \text{odd}(G - T) \leq |T|$$

If G is type 0 then G has a perfect matching.

Proof. Taking $T = X$ in $(\star\star)$ tells us that $\text{odd}(G - X) \leq |X|$, but $G - X = G[Y]$, so $G[Y]$ has at most $|X|$ odd component C. Begin constructing a matching by

- matching one vertex of each odd component of $G[Y]$ to a vertex of X. Match the rest of the vertices in each C inside C.
- matching the even components C can all be matched inside C.
- matching the rest of X inside X.

The only obstruction is that the rest of X is odd, but it can not happen since $V(G)$ is even by Remark 7.4. □

Proof of Tutte's Theorem (Lovász)

Proof. Suppose a graph G satisfies $(\star\star)$, but on the contrary does not have a perfect matching. By (possibly) adding edges one by one, construct a graph H with $V(H) = V(G)$ and such that H has no perfect matching but $H + e$ has a perfect matching for every $e \notin E(H)$. By Lemma 7.5, H satisfies $(\star\star)$. We will show H is type-0, contradicting Lemma 8.2.

Let $X \subseteq V(H)$ be the set of $x \in V(H)$ where $\Gamma(x) = V(H) \setminus \{x\}$ ($X = \emptyset$ is possible). Let $Y = V(H) \setminus X$. If $Y = \emptyset$, then H is type-0. Let C be a component of $G[Y]$. If $|V(C)| = 1$ or 2, then trivially complete and we are done. Suppose $|V(C)| \geq 3$.

Claim 8.3. If C is not a complete graph, then there exist $a, b, c \in V(C)$ with $ab, bc \in E(C)$ and $ac \notin E(G)$.

Proof. Since C is not complete, there exist a and x where $ax \notin E(C)$. Take a shortest path P from a to x in C and take b, c to be the second and the third vertices on P. □

Let $d \notin \{a, b, c\}$ where $bd \notin E(H)$. Such a d exists since $b \notin X$. By the property of H, $H + ac$ has a perfect matching M_1 and $H + bd$ has a perfect matching M_2. Then, the graph J with edge set $M_1 \cup M_2$ is a disjoint union of single edges (in $M_1 \cap M_2$) and (M_1, M_2)-alternating cycles. Since $ac \in M_1 \setminus M_2$, the component K of J containing ac is a cycle component. If $bd \notin K$, then the matching $M_1 \Delta E(K)$ is a perfect matching of H, so $bd \in K$ also. Then, there is a path P in K from b to (WLOG) a that does not contain d or c. Then $P \cup \{bd\} \cup \{da\}$ is an M_2-alternating cycle C', but then $M_2 \Delta E(C')$ is a perfect matching of H. This contradiction shows that H is type 0, which contradicts Lemma 8.2. □

Lecture 9

Application of Tutte's Theorem

9.1. Tutte's Theorem on Bipartite Graphs

Halls's Theorem implies Tutte's Theorem when G is bipartite.

Proof. Let G be a bipartite graph with vertex classes X and Y. Suppose that $(\star\star)$ for every $T \subseteq V(G)$, $\text{odd}(G-T) \leq |T|$. We show that G has a perfect matching (using Hall's Theorem).

- Taking $T = X$: then $G - T$ consists of $|Y|$ isolated vertices. Each vertex of Y has an odd component of $G - X$, so $\text{odd}(G-X) \geq |Y|$. Hence by $(\star\star)$, $|Y| \leq \text{odd}(G-X) \leq |X|$.
- Taking $T = Y$: similarly we obtain $|X| \leq |Y|$.

Hence, $|X| = |Y|$. To verify Hall's Condition, take an arbitrary subset $S \subseteq X$. Then the graph $G - \Gamma(S)$ has at least $|S|$ isolated vertices (the vertices in S), i.e.,

$$|\Gamma(S)| \underbrace{\geq}_{(\star\star)} \text{odd}(G-\Gamma(S)) \geq |S|$$

Hence, Hall's Condition holds, so G has a perfect matching. □

9.2. Peterson's Theorem

Lemma 9.1. *If G is a graph with $|V(G)|$ even, and $T \subseteq V(G)$, then* $\text{odd}(G-T) \equiv |T| \mod 2$.

Proof. Since $|V(G)|$ is even, $|T|$ and $|V(G-T)|$ have the same parity. Thus,

$$\text{odd}(G-T) \equiv |V(G-T)| \equiv |T| \mod 2$$

□

Definition 9.2 (Cut-Edge)**.** A cut-edge (or a bridge) of a connected graph is an edge e such that $G \setminus e$ is disconnected.

Theorem 9.3 (Peterson's Theorem)**.** *Let G be a connected* 3*-regular graph with at most* 2 *cut-edges. Then G has a perfect matching.*

Proof. Suppose G does not have a perfect matching, then by Tutte's Theorem, there exists $T \subseteq V(G)$ such that $\text{odd}(G - T) > |T|$. Let C be an odd component of $G - T$.

Claim 9.4. The number $|E(C)|$ of edges joining C to T is odd.

Proof.

$$\sum_{v \in C} d(v) = 2(E(G[C])) + |E(C, T)|$$

however,

$$\sum_{v \in C} d(v) = 3|V(C)|$$

which is odd. □

If $|E(C, T)| = 1$, then the edge in $E(C, T)$ is an cut-edge. Since G has at most 2 cut-edges. Hence, at least $\text{odd}(G - T) - 2$ odd components C have $|E(C, T)| \geq 3$. Then, the total number of edges from $V(G - T)$ to T is at least

$$3(\text{odd}(G - T) - 2) + 2$$

By Lemma 9.1, since $|V(G)|$ is even because G is 3-regular,

$$\text{odd}(G - T) \geq |T| + 2$$

Thus, we get at least $3|T| + 2$ edges from $G - T$ to T. This contradicts the fact that every vertex in T has degree 3. Hence, G admits a perfect matching.

□

Lecture 10

Defect Version of Tutte's Theorem and Edmonds Algorithm

10.1. Defect Version of Tutte's Theorem

Theorem 10.1 (Defect Version of Tutte's Theorem)**.** *Let G be a graph and let d be a non-negative integer such that $d \equiv |V(G)| \mod 2$. Then G has a matching saturating at least $|V(G)| - d$ vertices if and only if*

$$\text{odd}(G - T) \leq |T| + d \qquad \forall\ T \subseteq V(G)$$

Proof. Note that if $d = 0$, then this is Tutte's Theorem, so we may assume $d \geq 1$.

($\Rightarrow$): If G has a matching M saturating at least $|V(G)| - d$ vertices, and $T \subseteq V(G)$, then at most d of the odd components of $G - T$ contain an M-exposed vertex. Hence at least $\text{odd}(G - T) - d$ odd components have an edge of M joining it to T, so $|T| \geq \text{odd}(G - T) - d$.

($\Leftarrow$): Assume $\text{odd}(G - T) \leq |T| + d$ for each $T \subseteq V(G)$. Construct a graph H by adding a set A of d new vertices to G, and joining each $a \in A$ to all of $V(H) \setminus \{a\}$. Note $|V(H)|$ is even by the assumption $d \equiv |V(G)| \mod 2$. We show H has a perfect matching by verifying Tutte's Condition: Let $S \subseteq V(H)$.

- If $S \neq \emptyset \subseteq V(H)$ and $A \not\subseteq S$, then $\text{odd}(H - S) \leq 1$, since $H - S$ is connected via A. Hence, $\text{odd}(H - S) \leq 1 \leq |S|$.
- If $A \subseteq S$, then $\text{odd}(H - S) = \text{odd}(G - (S \setminus A))$, so $\text{odd}(H - S) = \text{odd}(G - (S \setminus A)) \leq |S \setminus A| + d = |S| - d + d = |S|$.
- If $S = \emptyset$, then $\text{odd}(H - S) = 0$ since H is connected via A and $|V(H)|$ is even.

By Tutte's Theorem, H has a perfect matching M.

Then, M saturates at least $|V(G)| - d$ vertices of G with edges of $M \cap E(G)$. □

10.2. Edmonds Algorithm

Definition 10.2. Let G be a graph and let M be a matching in G. Let C be an odd cycle in G with length $2k+1$. We say C is a shrinkable odd cycle with respect to M if exactly k of the edges of C are in the matching M, and C has one M-exposed vertex.

Note then that in the graph G' obtained by contracting all edges of C into a single vertex c, the vertex c is M'-exposed where $M' = M \setminus E(C)$.

Lemma 10.3 (Cycle Shrinking)**.** *Let M be a matching in G and let C be a shrinkable odd cycle with respect to M. Let G' be the graph obtained from G by contracting $E(C)$, and set $M' = M \setminus E(C)$. Then M is maximum in G if and only if M' is maximum in G'.*

Proof. ($\Leftarrow$): Assume that M' is maximum in G'. Suppose on the contrary that M is not maximum in G. Then by Theorem 3.7, G contains an M-augmenting path P. Then P intersects C, otherwise it is an M'-augmenting path in G'. Since C is shrinkable and P has 2 M-exposed vertices, but C has only one M-exposed vertex, one M-exposed endpoint z of P is not on C. Then the (z, c)-segment of P in G' is an M'-augmenting path, contradicting the maximality of M' in G'. □

Lecture 11

Edmonds Algorithm Cont'd

11.1. Cycle Shrinking

We will continue the proof of the Cycle Shrinking Lemma.

Lemma 11.1 (Cycle Shrinking)**.** *Let M be a matching in G and let C be a shrinkable odd cycle with respect to M. Let G' be the graph obtained from G by contracting $E(C)$, and set $M' = M \setminus E(C)$. Then M is maximum in G if and only if M' is maximum in G'. Let C be length $2k+1$.*

Proof. ($\Leftarrow$): Assume that M' is maximum in G'. Suppose on the contrary that M is not maximum in G. Then by Theorem 3.7, G contains an M-augmenting path P. Then P intersects C, otherwise it is an M'-augmenting path in G'. Since C is shrinkable and P has 2 M-exposed vertices, but C has only one M-exposed vertex, one M-exposed endpoint z of P is not on C. Then the (z, c)-segment of P in G' is an M'-augmenting path, contradicting the maximality of M' in G'.

($\Rightarrow$): Assume M is maximum in G. Suppose on the contrary that M' is not maximum in G'. Let N' be a matching in G' with $|N'| > |M'|$. Then each edge of N' corresponds to an edge in G, i.e., is either itself an edge of G or it corresponds to some edge G with one endpoint in cycle C. Then since at most one edge of M' is incident to a vertex of C, we can add k edges of C to N' to set a matching of size $|N'| + k > |M'| + k = |M|$. This contradicts the maximality of M. Hence M' is maximum in G. □

Remark 11.2. Note that

- The condition that C contains an M-exposed vertex is necessary.
- The proof does not say that if N' is a maximum matching in G', then we can get a maximum matching of G by adding k edges of C to M'. It only says that if $|N'| > |M'|$, then this gives a bigger matching than $|M|$ in G.

11.2. Alternating Forests

Ingredients of Edmonds Algorithm

- Cycle Shrinking
- Alternating Forests

Edmonds Algorithm constructs a special subgraph in a graph G with a given matching M. This special subgraph is called an M-alternating forest.

Definition 11.3 (Inner/Outer Vertices)**.** The vertices at T at an odd distance from s in T are called the inner vertices of T. The rest including s are called outer vertices.

Definition 11.4 (M-Alternating Forest)**.** An M-alternating forest is any subgraph whose components are such that

- T contains exactly one M-exposed vertex s,
- Every edge of T at an odd distance from s in T is in M.
- Every vertex at an odd distance from s in T has degree 2 in T.

Definition 11.5 (Maximal M-Alternating Forest)**.** Let F be an M-alternating forest. We say F is a maximal M-alternating forest if it is not contained in any strictly larger (i.e. more vertices) M-alternating forest.

Lecture 12

Alternating Forests and the Maximum Matching

`[Insert Example Here]`

Remark 12.1. An M-alternating forest is maximal if it contains all the vertices of a graph.

How to recognize that M is a maximum matching from an M-alternating forest?

Lemma 12.2. *Let G be a graph and let H be a matching in G. Let F be a maximal M-alternating forest in G. Suppose there is no edge of G joining two outer vertices of F. Then M is a maximum matching in G.*

Proof. Let F be a maximal M-alternating forest as in the statement. The following remarks are useful in the proof:

Remark 12.3. Every M-exposed vertex of G is in F, since F is maximal.

Remark 12.4. If $xy \in E(M)$ and x is in some component T of F, then y is also in T.

Claim 12.5. If z is an outer vertex of F then $\Gamma_G(z)$ is contained in the set of inner vertices of F.

Proof. To see this, let $y \in \Gamma_G(z)$. Then y is not an outer vertex of F by assumption. Suppose $y \notin V(F)$. By Remark, 12.3 y is not M-exposed, so $xy \in M$ for some x. Then, by Remark 12.4, $x \notin V(F)$. But then, F is not maximal, since y and x could have been added to the same component of F that contains z. Hence, y must be an inner vertex. □

Let A_T and B_T denote the sets of inner and outer vertices of component T of F respectively.

Let $A = \bigcup_{T \text{ component of } F} A_T$ and $B = \bigcup_{T \text{ component of } F} B_T$ be the sets of inner and outer vertices of F. Hence, $F_G(B) \subseteq A$.

Then for each component T of F we have

$$|A_T| + 1 = |B_T|$$

so $|B| = |A| + c$ where c is the number of components of F. Note that c is the number of M-exposed vertices in G.

Recall from the defect version of Tutte's Theorem that for any subset $S \subseteq V(G)$, in each odd component of $G - S$ there must be a vertex that is either M-exposed or matched by M to a vertex of S.

Consider $A \subseteq V(G)$. The number of odd components of $G - A$ is at least $|B| = |A| + c$, since each vertex of B is a component of size 1 in $G - A$ by the claim. Thus, $\text{odd}(G - A) \geq |A| + c$, but the number of M-exposed vertices is exactly c, so M is a maximum matching in G. □

Lecture 13

Overview of Edmonds Algorithm

We will now address the question of what to do when we find edges between two outer vertices.

Lemma 13.1. *Let G be a graph. Let M be a matching in G and let F be an M-alternating forest in G. Suppose there exists an edge of G joining two outer vertices of F that are in distinct components of F. Then G contains an M-augmenting path.*

Proof. Let xy be the edge. Then the path, from the M-exposed vertex in the component T_x of F containing x to x, together with xy and the corresponding path in T_y from y to its M-exposed vertex, is M-augmenting. □

Lemma 13.2. *Let G be a graph. Let M be a matching in G and let F be an M-alternating forest in G. Suppose there exists an edge e of G joining two outer vertices of F that are in the same component of F. There exists a matching $\overline{M}$ in G with $|\overline{M}| = |M|$ and an odd cycle C that is shrinkable with respect to $\overline{M}$.*

Proof. Let C be the odd cycle formed by adding the edge e to the component T of F containing both its endpoints. Let P denote the path in T from C to the M-exposed root of T. Set $\overline{M} = M \Delta E(P)$. Then $|\overline{M}| = |M|$ and C is shrinkable with respect to $\overline{M}$. □

Idea of Edmonds Algorithm

1. Start with a graph G and a matching M in G. Aim to either certify that M is maximum or find a matching of size greater than $|M|$.

2. Start constructing an M-alternating forest in G.

3. If you find an edge of G joining two outer vertices of F that are in the same component,

   - switch M to $\overline{M}$,
   - shrink the shrinkable odd cycle C to get $G' = G/E(C)$, and
   - recursively run the algorithm on G' and $M' = \overline{M} \setminus E(C)$. *This either certifies M' is maximum in G', or finds a larger matching in G', leading to a larger matching in G.*

4. If you find an edge of G joining two outer vertices of F that are in different components of F, then find an M-augmenting path. Switch on it to get a larger matching. Start over.

5. If you never find an edge joining two outer vertices after completing the construction of a maximal M-alternating forest, then M is maximum. Stop. Output M.

Example 13.3. `Place example here.`

Lecture 14

Finalizing Edmonds Algorithm

14.1. Edmonds Algorithm

INPUT: A graph G and a matching M.
Maybe start with a greedy matching to save some steps.
OUTPUT: A maximum matching M^* in G.

1. Set $G_0 := G$ and $M_0 := M$.
2. Let S be the set $\{s_1, \ldots, s_r\}$ of M-exposed vertices.
3. Set outer $:= S$, inner $:= \emptyset$, checked $:= \emptyset$, $F := \emptyset$ (*edges of M-alternating forest*). Label each $s_i \in$ outer with $\ell(s_i) := i$.
4. Choose $v \in$ outer and edge $vw \notin$ checked if they exist. If none exists then M_0 is maximum in G_0. Set $M^* := M_0$. Stop.
5. If $w \in$ inner, add $wv \in$ checked, and goto 4.
6. If $w \in$ outer and $\ell(w) = \ell(v)$ (*using label to check whether they are in the same component*) then the cycle C in $F \cup \{vw\}$ is shrinkable with respect to $\overline{M}$ as in Lemma 13.2. Record the subgraph induced by $V(C)$, and the edges joining $V(C)$ to the rest of C. Shrink C: replace G by $G/E(C)$ and M by $\overline{M} \setminus E(C)$, and goto 2.
7. If $w \in$ outer and $\ell(w) \neq \ell(v)$ then there is an M-augmenting path P by Lemma 13.1. Switch on P to set a bigger matching. Expand all shrunken cycles and add k edges in each $(2k+1)$-cycle to the matching. Replace M by this new bigger matching of G by the original graph, and goto 1.
8. If $w \notin$ inner $\cup$ outer, then there exists $wx \in M_0$ (*expanding the M-alternating fortest*), add vw and wx to F and to checked. Add w to inner, x to outer. Set $\ell(w) = \ell(x) = \ell(v)$. (*$w, x, v$ are in the same component.*)

Lecture 15

f-Factor

Definition 15.1 (f-Factor)**.** Let G a graph and let $f : V(G) \to \{0, 1, 2, \ldots\}$ be a function. An f-factor in G is a subgraph H of G such that $d_H(v) = f(v)$ for all $v \in V(G)$.

`[Example]`

Example 15.2. A perfect matching in G is a $\mathbb{1}$-factor, where $\mathbb{1}$ denotes the function that assigns 1 each vertex $v \in V(G)$.

When f is a constant function, we often write k-factor, where $f(v) = k \; \forall v \in V(G)$.

Reduction from f-Factor Problem to $\mathbb{1}$-Factor Problem

Let G be a graph and f a function on $V(G)$.

Remark 15.3. If $f(v) > d(v)$ for some v then clearly G has no f-factor.

We will reduce the problem of finding an f-factor in G (and in particular, determining if one exists) to the same problem in the following auxiliary graph.

Definition 15.4 ($H(G, f)$)**.** Define $H(G, f)$:

- For each $v \in V(G)$, we take sets $A(v)$ and $B(v)$ of vertices where $|A(v)| = d(v)$ and $|B(v)| = d(v) - f(v)$.
- All of $A(v)$ is joined to all of $B(v)$ for each v.
- For each edge xy of G, we put one edge from $A(x)$ to $A(y)$ in $H(G, f)$ so that all of these edges are vertex-disjoint.

Theorem 15.5 (Tutte)**.** *A graph G has an f-factor if and only if $H(G, f)$ has a perfect matching.*

Proof. ($\Rightarrow$): First suppose G has an f-factor J. Then the edges of J correspond to a matching M in $H(G, f)$ such that exactly $f(v)$ vertices of $A(v)$ are saturated by M. Then M leads to a prefect matching of $H(G, f)$ by matching the remaining $d(v) - f(v)$ vertices in $A(v)$ to $B(v)$.

($\Leftarrow$): Suppose H has a perfect matching M. Then the vertices in $B(v)$ must be matched to vertices in exactly $|B(v)| = d(v) - f(v)$ vertices in $A(v)$ for each v. Then M saturates exactly $f(v)$ vertices of $A(v)$ via edges that correspond to edges of G. Thus these edges form an f-factor of G. □

15.1. Eulerian Circuits

Definition 15.6 (Eulerian Circuit)**.** A sequence $v_0 e_1 v_1 e_2 v_2 \cdots e_n v_n$ of vertices and edges in a graph G is an Eulerian circuit if

- $v_0 = v_n$
- $e_i = v_i v_{i-1}$ for each i appears exactly one
- Each $e \in E(G)$

Definition 15.7. A graph is even if every vertex has even degree.

Since each visit of an Eulerian circuit to a vertex v uses 2 edges incident to v, if G has an Eulerian circuit then it is even. Also G must be connected to contain an Eulerian circuit.

Theorem 15.8. *A connected graph G has an Eulerian circuit if and only if it is even.*

Proof. ($\Rightarrow$): Discussed above.

($\Leftarrow$): Assume G is even. We will prove by induction on $E(G)$ that it has an Eulerian circuit.

- Base case: $m = |E(G)| = 0$. Then $V(G) = \{x\}$, which is an Eulerian circuit with 0 edges.
- Inductive hypothesis: Assume $m \geq 1$ and every connected even graph with fewer than m edges has an Eulerian circuit.

- We may assume $m \geq 1$ and G is an even graph with m vertices. Let v_0 be an arbitrary vertex and let $W = v_0 e_1 v_1 e_2 v_2 \cdots e_r v_r$ be a maximal sequence such that each $e_i = v_{i-1} v_i$ for each i and all the e_i are distinct. Then since G is even, then $v_r = v_0$ (Note that W contributes 2 to every intermediate vertex, and hence also to $v_0 = v_r$.) If $\{e_1, \ldots, e_r\} = E(G)$, then W is an Eulerian circuit. Otherwise, consider G' where $V(G') = V(G)$ and $E(G') = E(G) \setminus \{e_1, \ldots, e_r\}$. Then since G is even, G' is also even. Thus, by induction hypothesis, every component G_i of G' has an Eulerian circuit T_i. Since G is connected, every G_i that is not an isolated vertex shares a vertex a_i with W, so we may insert T_i at a_i into W for each i to get an Eulerian circuit of G.

□

Lecture 16

Regular Graph and 2-Factor

Definition 16.1. For a positive integer k, a k-factor in a graph G is a spanning subgraph H such that $d_H(V) = k$ for each $v \in V(G)$.

Theorem 16.2 (Peterson)**.** *Every r-regular graph, where $r \geq 2$ is even, has a 2-factor.*

Proof. We may assume that G is a connected r-regular graph by considering each component separately. Then G has an Eulerian circuit C.

Construct a bipartite graph J with vertex classes A and B, both of which are copies of $V(G)$. We join $v \in A$ to $w \in B$ for each edge vw such that C uses vw in the direction $v \to w$, so every edge of G is represented exactly once in J. Then each vertex of B has degree $r/2$. Hence by our corollary of Hall's Theorem, J has a perfect matching M. Each edge of M is an edge of G, so M represents $|V(G)|$ edges in G. The copy of v in A is incident to 1 edge of M and the copy of v in B is incident to 1 edge of M, and these are different edges by construction. Hence, M is a 2-factor in G. $\square$